תשע"ב ב'

1) $Q(x,y) \Leftrightarrow y=2x$.

$\forall x \in \mathbb{N}\ \exists y \in 2\mathbb{N} : y=2x \Rightarrow Q(x,y)$

$\forall y \in 2\mathbb{N}\ \exists x \in \mathbb{N} : y=2x \Rightarrow Q(x,y)$

$\nexists y \in 2\mathbb{N} : 2y=3 \Rightarrow \neg Q(y,x)$

2) $a \sim b \Leftrightarrow \frac{a}{b} \in \mathbb{Q} \quad \forall a,b \in \mathbb{R}\setminus\{0\}=\mathbb{R}^*$

א. $\forall a \in \mathbb{R}^* \quad \frac{a}{a}=1 \in \mathbb{Q} \Rightarrow a \sim a$

$a \sim b \Rightarrow \frac{a}{b} \in \mathbb{Q} \Rightarrow \frac{b}{a} \in \mathbb{Q} \Rightarrow b \sim a$

$a \sim b \sim c \Rightarrow \frac{a}{b}, \frac{b}{c} \in \mathbb{Q} \Rightarrow \frac{a}{b}\frac{b}{c} = \frac{a}{c} \in \mathbb{Q} \Rightarrow a \sim c$

ב. $\bar{a}=\{b : \frac{a}{b} \in \mathbb{Q}\} \quad a,b \in \mathbb{R}^*, \frac{a}{b} \in \mathbb{Q} \Rightarrow a=a'c',\ b=b'c'$

כך ש- $a',b' \in \mathbb{Q}$ ו- $c' \in \mathbb{R}^*\setminus\mathbb{Q}$ משותף. לכן:

$\bar{1} = \mathbb{Q}\setminus\{0\}$

$\forall a \in \mathbb{R}^*\setminus\mathbb{Q} \quad \bar{a} = \{a \cdot x : x \in \mathbb{Q}^*\setminus\{0\}\}$

⇐ כל מחלקת שקילות שווה עוצמה ל-$\mathbb{Q}$

ולכן בת מניה.

ג. [redacted]

נסמן: $\mathbb{R}^*/\sim = \{\bar{1}\} \cup \{\bar{a} : a \in \mathbb{R}^*\setminus\mathbb{Q}\}$

ונקח המייצגים הנ"ל:

$A = \{1\} \cup \{a : \bar{a} \in \mathbb{R}^*/\sim\}$

$P=\{2,3,4\}\times\{1,3,5\}$ (3

$(x_1,y_1)\precsim(x_2,y_2) \Leftrightarrow (x_1|x_2)\wedge(x_1+y_1\le x_2+y_2)$

✓ $\forall (x,y)\in P \quad (x|x)\wedge(x+y\le x+y)\Rightarrow (x,y)\precsim(x,y)$ .א

✓ $(x_1,y_1)\precsim(x_2,y_2)\wedge(x_2,y_2)\precsim(x_1,y_1)\Rightarrow$

$(x_1|x_2)\wedge(x_2|x_1)\wedge(x_1+y_1\le x_2+y_2)\wedge(x_2+y_2\le x_1+y_1)\Rightarrow$

$x_1=x_2 \wedge y_1=y_2$

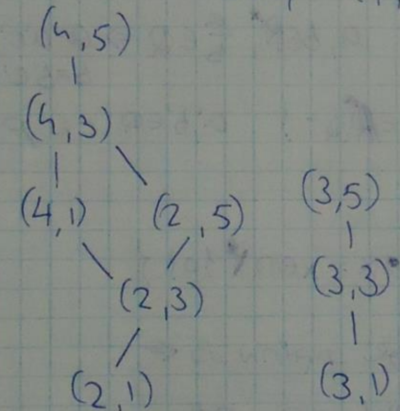
✓ $(x_1,y_1)\precsim(x_2,y_2)\precsim(x_3,y_3)\Rightarrow$

$(x_1|x_2)\wedge(x_2|x_3)\wedge(x_1+y_1\le x_2+y_2\le x_3+y_3)\Rightarrow$

$(x_1|x_3)\wedge(x_1+y_1\le x_3+y_3)\Rightarrow(x_1,y_1)\precsim(x_3,y_3)$

$\Leftarrow (P,\precsim)$ יס"ח.

.ב




.ג :שרשרת

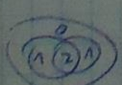
$(2,1)\precsim(2,3)\precsim(2,5)\precsim(4,3)\precsim(4,5)$

$(2,1)\precsim(2,3)\precsim(4,1)\precsim(4,3)\precsim(4,5)$

ד

max: $(4,5)$, $(3,5)$

min: $(2,1)$, $(3,1)$



.א (4

$\forall x\in X \quad \chi_A=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$

$\chi_B=\begin{cases}1, & x\in B\\ 0, & x\notin B\end{cases}$

$\chi_A+\chi_B=\begin{cases}1, & x\in(A\setminus B)\cup(B\setminus A)\\ 0, & x\notin A\cup B\\ 2, & x\in A\cap B\end{cases}$

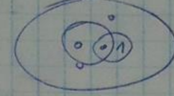
:רק

$\Rightarrow \chi_A+\chi_B=\begin{cases}1, & x\in A\cup B\\ 0, & x\notin A\cup B\end{cases} \Leftrightarrow A\cap B=\emptyset$

ב.

$$(1-\chi_A)\chi_B = \left(\begin{cases}0, & x\in A\\ 1, & x\notin A\end{cases}\right)\left(\begin{cases}1, & x\in B\\ 0, & x\notin B\end{cases}\right) = \begin{cases}0 & , x\in A\setminus B\\ 0 & , x\notin A\cup B\\ 0 & , x\in A\cap B\\ 1 & , x\in (B\setminus A)\end{cases}$$

$$\left(\begin{cases}1, & x\in A^c\\ 0, & x\notin A^c\end{cases}\right)\left(\begin{cases}1, & x\in B\\ 0, & x\notin B\end{cases}\right)$$



$$\Rightarrow (1-\chi_A)\chi_B = \begin{cases}1, & x\in B\setminus A\\ 0, & x\notin B\setminus A\end{cases} = \chi_{B\setminus A}$$

$$|A^A| \neq \aleph_0$$

5ב. צ"ל:

I: $n = |A| < \aleph_0 \Rightarrow$ סופית $A \Rightarrow A^A$ סופית $\Rightarrow |A^A| \neq \aleph_0$

$$|\underbrace{A\times A\times\cdots\times A}_{n \text{ פעמים}}| = \underbrace{|A|\cdots|A|}_{n \text{ פעמים}} = |A|^n = n^n$$

II: $|A| = \aleph_0 \Rightarrow |A^A| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph > \aleph_0$

III: $|A| \geq \aleph \Rightarrow \aleph^{\aleph} = 2^{\aleph} > \aleph > \aleph_0$

ג.

$f: A \longrightarrow A$

$f^4 = Id$

$f^4(x) = x$

$$f(x_1) = f(x_2)$$
$$\Downarrow$$
$$f^3(f(x_1)) = f^3(f(x_2))$$
$$\Downarrow$$
$$f^4(x_1) = f(x_2)$$
$$\Downarrow$$
$$\boxed{f \text{ חח"ע}} \Leftarrow x_1 = x_2$$

$$A = Im(Id) = Im(f^4) \subseteq Im(f^3) \subseteq Im(f^2) \subseteq Im(f) \subseteq A \ (*)$$

$(*)$ $y \in Im(f^4) \Rightarrow \exists x: f^4(x) = y$

$\Rightarrow f^3(f(x)) = y,\ f^2(f^2(x)) = y,\ f(f^3(x)) = y$

$$\Rightarrow Im(f) = A \Rightarrow \boxed{f \text{ על}}$$